長崎総合科学大学 紀 要 第 63 巻 第 2 号（pp. 117～143）

【 数物系科学 】
〈 研究ノート 〉

Proof that the Center of Buoyancy is Equal to the Center of Hydrostatic Pressure
— Part 2 : Semi-Submerged Circular Cylinder and Triangular Prism —

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Summary

We recently proved that "the center of buoyancy of floating bodies is equal to the center of hydrostatic pressure". This subject was an unsolved problem in physics and naval architecture, even though the buoyancy taught by Archimedes' principle can be obtained clearly by the surface integral of hydrostatic pressure. Then we thought that the reason why the vertical position of the center of pressure could not be determined was that the horizontal force would be zero due to equilibrium in the upright state.

As a breakthrough, we dared to assume the left-right asymmetric pressure field by inclining the floating body with heel angle θ. In that state, the force and moment due to hydrostatic pressure were calculated correctly with respect to the tilted coordinate system fixed to the body. By doing so, we succeeded in determining the center of pressure. Then, by setting the heel angle θ to zero in order to make it upright state, it could be proved that the center of hydrostatic pressure is equal to the well-known center of buoyancy, *i.e.*, the centroid of the cross-sectional area under the water surface.

As noted above, we have already proved this problem for rectangular and arbitrarily shaped cross-sections, and published them here on *viXra.org* in English. Although the case of a semi-submerged circular cylinder and a triangular prism are also included in the proof of arbitrary shapes, we prove for each shape separately in this 2nd report, since they are two typical cross-sectional shapes along with rectangles. However, there is an essential difference in the proof between the two shapes. The reason is why the former does not change its underwater shape when inclined laterally, while the latter, like the rectangle, changes its cross-sectional shape when inclined. The present paper provides clear proofs for both shapes.

Keywords : ***Center of Buoyancy, Hydrostatic Pressure, Archimedes' Principle, Surface Integral, Semi-Submerged Circular Cylinder, Triangular Prism***

1. Introduction

It is a well-known fact in physics and naval architecture that the position of "*Center of Buoyancy*" acting on a ship is equal to the center of the volume of the geometric shape under the water surface.

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Received : September 28, 2023 《 The Bulletin of Nagasaki Institute of Applied Science, Vol.63, No.2,
Accepted : October 23, 2023 Research Notes in Mathematical and Physical Science 》

The buoyancy taught by Archimedes' principle (1) is clearly obtained by the surface integral of the hydrostatic pressure, but the position of the center of buoyancy is described in every textbook (on physics (2), fluid dynamics (3),(4), hydraulics (5), naval architecture (6),(7),(8),(9),(10),(11) and nautical mechanics (12), *etc.*) as the center of gravity where the volume under the water surface is replaced by water. There is no explanation that it is the center of pressure due to hydrostatic pressure (13),(14).

Recently, Komatsu (15) raised the issue of "the center of buoyancy ≠ the center of pressure ? " at 2007 in Japan, and it was actively discussed by Seto (16),(17), Suzuki (18), Yoshimura and Yasukawa (19), Komatsu (20), Yabushita and Watanabe (21) and others in research committees and academic meetings of the Japan Society of Naval Architects and Ocean Engineers (hereinafter abbreviated as *JASNAOE*). At the same time, in Europe, the problem was studied in detail by Mégel and Kliava (22),(23) in terms of potential energy. However, no one was able to solve this issue.

On the other hand, it is also an indisputable fact that the well-known center of buoyancy (*i.e.* the volume center of the underwater portion) is correct from the viewpoint of ship's hydrostatic stability (24)~(29) (that is to say, positioning of the metacenter by calculating the metacentric radius (30)~(33) $\overline{BM}$).

In response to this unsolved problem, we considered that the reason why the vertical center of pressure could not be determined was because the horizontal forces equilibrated to zero in the upright state. To solve this problem, Hori (34),(35) attempted in 2018 to integrate the hydrostatic pressure acting on the ship surface at the inclined state with heel angle θ. Then, the forces and moments acting on the ship were calculated with respect to a tilted coordinate system fixed to the ship. In this case, both orthogonal components of the force acting on the ship are not zero. Therefore, it was shown that the center of pressure at the inclined state can be determined. By setting the heel angle θ to zero, we proved that the center of hydrostatic pressure coincides with the centroid of cross-sectional area under the water surface in the upright state, *i.e.*, the well-known center of buoyancy. First, a columnar ship with the rectangular cross-section (34) was proved and its proof is lectured (36),(37) to 2nd year students of the naval architectural engineering course (38),(39) in the " *Hydrostatics of Floating Bodies* " of the university where one of the authors (40) works. And then an arbitrary cross-sectional shape (35) was proved and published in the Journal "*NAVIGATION*" of Japan Institute of Navigation (hereinafter abbreviated as *JIN*).

In other way, as many researchers are studying this issue with various approaches (41)~(45), the discussions have deepened in *JASNAOE*. To sublate these discussions, we have illustrated that " the center of buoyancy is equal to the center of pressure " for a semi-submerged circular cylinder 1st half of (46) and a submerged circular cylinder (47) which does not change its shape under the water even if it is inclined, and for a triangular prisms (48), using the same method (49).

In order to put an end to the above discussions, we proved that " the center of buoyancy = the center of pressure" for a submerged body with arbitrary shape 1st half of (50) using Gauss's integral theorem in 2021. Furthermore, it was published in the same journal "*NAVIGATION*" of *JIN* that it is easier to prove for a floating body with arbitrary shape 2nd half of (50) than author's previous paper (35) by using Gauss's theorem in the same way (51).

We subsequently summarized the proofs in English for the case of the rectangular cross-section [34], which is the easiest to understand, and for the floating body of arbitrary cross-sectional shape [2nd half of (50)] by applying Gauss's integral theorem. And we published them on this *viXra.org* [52] and in the bulletin of our university, *Nagasaki Institute of Applied Science* [53]. Furthermore, we showed an extension to the center of buoyancy for a 3-D floating body. More recently, the authors have summarized the above as a new developments for the fundamental theory of hydrostatics of floating body and published it here on viXra.org [54].

As noted above, we have already proved this problem for rectangular and arbitrarily shaped cross-sections and published it on the bulletin of our university [53] in English. Although the case of a semi-submerged circular cylinder and a triangular prism are also included in the proof of arbitrary shapes [35],[49]~[54], we prove for each shape separately in this 2nd report, since they are two typical cross-sectional shapes along with rectangles. However, there is an essential difference in the proof between the two shapes. The reason is why the former does not change its underwater shape when inclined laterally, while the latter, like the rectangle [34],[36],[37],[49],[52]~[54], changes its cross-sectional shape when inclined. The present paper provides clear proofs for both shapes [55].

We would like to report all of you smart readers about the two proofs.

2. Positioning of the Center of Hydrostatic Pressure C_P Acting on the Semi-Submerged Circular Cylinder

Fig. 2.1 shows that a cross-section of semi-submerged circular cylinder with radius R (breadth $2R$ and draft R) inclines laterally with a heel angle θ to the starboard side. The origin o is placed at the center of the still water surface, and the coordinate system fixed in space with the z-axis pointing vertically downward is $o-yz$, and that fixed to the inclined circular cylinder is $o-\eta\zeta$.

If the argument measured counterclockwise from the ζ - axis is ϕ as shown in Fig. 2.1, then the argument of the water surface on the port side ϕ_L and on the starboard side ϕ_R can be written respectively, as follows :

$$\left.\begin{aligned} \phi_L &= -\frac{\pi}{2}+\theta \\ \phi_R &= \frac{\pi}{2}+\theta \end{aligned}\right\} \qquad \cdots\cdots (2.1)$$

Here, the aerial part C_{air} and the submerged part C_{water} can be written in terms of argument ϕ, respectively, as follows :

$$\left.\begin{aligned} C_{air}:&\quad \phi_R \le \phi \le \phi_L+2\pi \;\rightarrow\; \frac{\pi}{2}+\theta \le \phi \le \frac{3\pi}{2}+\theta \\ C_{water}:&\quad \phi_L \le \phi \le \phi_R \;\rightarrow\; -\frac{\pi}{2}+\theta \le \phi \le \frac{\pi}{2}+\theta \end{aligned}\right\} \qquad \cdots\cdots (2.2)$$

The water depth $z(\phi)$ on the cylinder surface $(\eta,\zeta)=(R\sin\phi, R\cos\phi)$ is then obtained as :

$$z(\phi) = (\zeta + \eta \tan\theta)\cos\theta$$
$$= R(\cos\phi\cos\theta + \sin\phi\sin\theta)$$
$$= R\cos(\phi - \theta) \qquad (2.3)$$

Here, the notation in the 3rd line of the above equation is evident from Fig. 2.1.

And in the figure, the outward unit normal vector $\boldsymbol{n}$, standing on the cylinder surface, can be written using the argument ϕ, as follows :

$$\boldsymbol{n} = n_\eta \boldsymbol{j} + n_\zeta \boldsymbol{k}$$
$$= \sin\phi\,\boldsymbol{j} + \cos\phi\,\boldsymbol{k} \qquad (2.4)$$

Here, n_η and n_ζ are the directional cosines in the η and ζ coordinates fixed to the cylinder, and $\boldsymbol{j}$ and $\boldsymbol{k}$ are the basic vectors in the η and ζ directions, similarly.

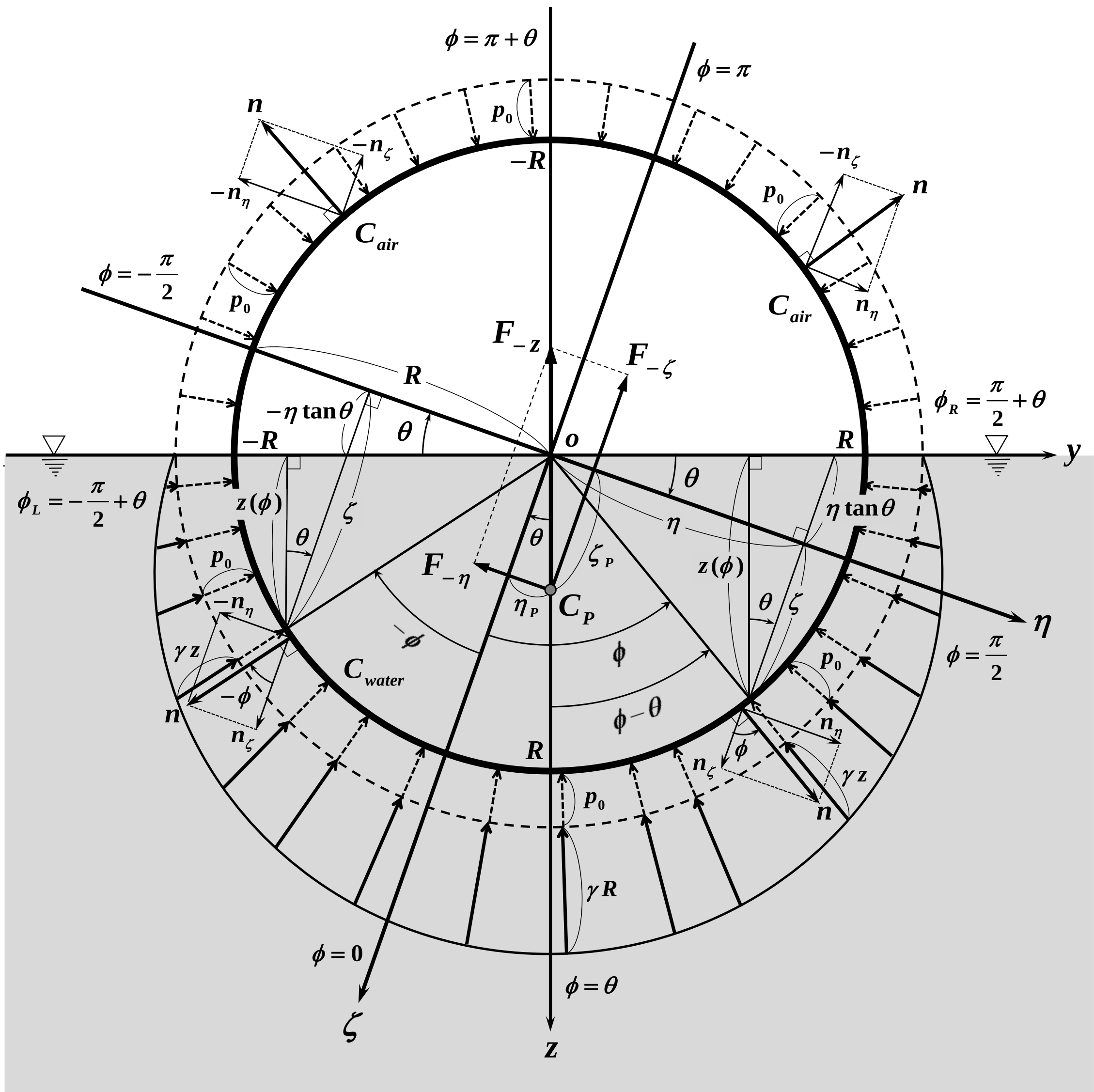

Fig. 2.1 Hydrostatic pressure and the center of pressure acting on the cross-section of an inclined semi-submerged circular cylinder.

In Fig. 2. 1, atmospheric pressure p_0 is shown as a dashed vector and hydrostatic pressure γz as a solid vector, and all are acting on $-\boldsymbol{n}$ direction perpendicular to the cylinder surface. Here, γ is the specific gravity of water.

2. 1 Forces $F_{-\eta}$ and $F_{-\zeta}$ due to pressure in the $-\eta$ and $-\zeta$ directions acting on the cylinder surface

Writing the total force acting on the cylinder as $\boldsymbol{F} = F_{-\eta}(-\boldsymbol{j}) + F_{-\zeta}(-\boldsymbol{k})$, the force $F_{-\eta}$ in $-\eta$ direction and the force $F_{-\zeta}$ in the $-\zeta$ direction due to pressure p acting on the cylinder surface are the sum of the force due to atmospheric pressure p_0 acting in the aerial part C_{air} and the force due to hydrostatic pressure $p_0+\gamma z$ acting in the submerged part C_{water} respectively, and are obtained as follows :

$$\left.\begin{aligned} F_{-\eta} &= \oint_{C_{air}+C_{water}} p\, n_\eta\, d\ell = \int_{C_{air}} p_0 n_\eta\, d\ell + \int_{C_{water}} (p_0+\gamma z)\, n_\eta\, d\ell \\ F_{-\zeta} &= \oint_{C_{air}+C_{water}} p\, n_\zeta\, d\ell = \int_{C_{air}} p_0 n_\zeta\, d\ell + \int_{C_{water}} (p_0+\gamma z)\, n_\zeta\, d\ell \end{aligned}\right\} \quad \cdots\cdots\cdots (2.5)$$

Here, on the cylinder surface ($\sqrt{\eta^2+\zeta^2}=R$), the line element is $d\ell = R\,d\phi$, and the directional cosines in the η and ζ directions can be written as $n_\eta=\sin\phi,\ n_\zeta=\cos\phi$ according to Eq. (2.4), so that for each part of C_{air} and C_{water}, both $F_{-\eta}$ and $F_{-\zeta}$ can be expressed by integration with respect to the argument ϕ in the interval of Eq. (2.2).

Therefore, $F_{-\eta}$ acting in the $-\eta$ direction is expressed as :

$$\begin{aligned} F_{-\eta} &= \int_{\frac{\pi}{2}+\theta}^{\frac{3\pi}{2}+\theta} p_0 \sin\phi\cdot R\,d\phi + \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} (p_0+\gamma z)\sin\phi\cdot R\,d\phi \\ &= p_0 R\int_{-\frac{\pi}{2}+\theta}^{\frac{3\pi}{2}+\theta} \sin\phi\, d\phi + \gamma R\int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} z\sin\phi\, d\phi \\ &= \gamma R\int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} z(\phi)\sin\phi\, d\phi \qquad \cdots\cdots\cdots (2.6) \end{aligned}$$

Similarly, $F_{-\zeta}$ acting in the $-\zeta$ direction is expressed as :

$$\begin{aligned} F_{-\zeta} &= \int_{\frac{\pi}{2}+\theta}^{\frac{3\pi}{2}+\theta} p_0 \cos\phi\cdot R\,d\phi + \int_{-\frac{\pi}{2}+\phi}^{\frac{\pi}{2}+\theta} (p_0+\gamma z)\cos\phi\cdot R\,d\phi \\ &= p_0 R\int_{-\frac{\pi}{2}+\theta}^{\frac{3\pi}{2}+\theta} \cos\phi\, d\phi + \gamma R\int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} z\cos\phi\, d\phi \\ &= \gamma R\int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} z(\phi)\cos\phi\, d\phi \qquad \cdots\cdots\cdots (2.7) \end{aligned}$$

The results of the above equations for both $F_{-\eta}$ and $F_{-\zeta}$ show that the integral over the entire circumference of cylinder with respect to the atmospheric pressure p_0 in the 1st term of 2nd line is zero and does not contribute to the force. Therefore, we can calculate only the 2nd term by using Eq. (2.3) for the water depth $z(\phi)$, so that $F_{-\eta}$ is obtained as :

$$
\begin{aligned}
F_{-\eta} &= \gamma R^2 \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} (\cos\theta\cos\phi + \sin\theta\sin\phi)\sin\phi\, d\phi \\
&= \gamma R^2 \left\{ \cos\theta \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \sin\phi\cos\phi\, d\phi + \sin\theta \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \sin^2\phi\, d\phi \right\} \\
&= \frac{1}{2}\gamma R^2 \left\{ \cos\theta \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \sin 2\phi\, d\phi + \sin\theta \left(\int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} d\phi - \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \cos 2\phi\, d\phi \right) \right\} \\
&= \gamma \frac{\pi R^2}{2} \sin\theta \qquad \cdots\cdots (2.8)
\end{aligned}
$$

And, $F_{-\zeta}$ is obtained as :

$$
\begin{aligned}
F_{-\zeta} &= \gamma R^2 \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} (\sin\theta\sin\phi + \cos\theta\cos\phi)\cos\phi\, d\phi \\
&= \gamma R^2 \left\{ \sin\theta \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \sin\phi\cos\phi\, d\phi + \cos\theta \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \cos^2\phi\, d\phi \right\} \\
&= \frac{1}{2}\gamma R^2 \left\{ \sin\theta \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \sin 2\phi\, d\phi + \cos\theta \left(\int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} d\phi + \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \cos 2\phi\, d\phi \right) \right\} \\
&= \gamma \frac{\pi R^2}{2} \cos\theta \qquad \cdots\cdots (2.9)
\end{aligned}
$$

These are because the integrals of $\sin 2\phi$ and $\cos 2\phi$ are zero, and the integral value of 2nd term is π in the 3rd line of both equations above. Both results indicate that $F_{-\eta}$ and $F_{-\zeta}$ are obtained as $-\eta$ and $-\zeta$ directional components of the buoyancy $\gamma \frac{\pi R^2}{2}$, as shown by F_{-z} of Eq. (2.10) in the next section.

2.2 Forces F_{-y} and F_{-z} converted in the $-y$ and $-z$ directions

By using $F_{-\eta}$ and $F_{-\zeta}$ obtained in Eqs. (2.8) and (2.9) of the previous section, the horizontal component F_{-y} and the vertical component F_{-z} are converted as follows :

$$
\left.
\begin{aligned}
F_{-y} &= F_{-\eta}\cos\theta - F_{-\zeta}\sin\theta \\
&= \gamma \frac{\pi R^2}{2} (\sin\theta\cdot\cos\theta - \cos\theta\cdot\sin\theta) = 0 \\
F_{-z} &= F_{-\zeta}\cos\theta + F_{-\eta}\sin\theta \\
&= \gamma \frac{\pi R^2}{2} (\cos^2\theta + \sin^2\theta) = \gamma \frac{\pi R^2}{2}
\end{aligned}
\right\} \qquad \cdots\cdots (2.10)
$$

The above results show that the horizontal component F_{-y} does not act as a combined force due to pressure integration. The vertical component F_{-z} is the product of the specific gravity γ of water and the area $\frac{\pi R^2}{2}$ of semicircle below the water surface, and is indeed buoyant force itself acted vertically upward, as Archimedes' principle(1) teaches.

2.3 Moments M_η and M_ζ due to pressure in the η and ζ directions acting on the cylinder surface

The clockwise moment M_η about the origin o due to the pressure p in the $-\eta$ direction acting on the cylinder surface and the counterclockwise moment M_ζ due to the pressure in the $-\zeta$ direction can be obtained by integrating Eq. (2.5) multiplied by ζ or η as the lever of the moment respectively, as follows :

$$\left.\begin{aligned} M_\eta &= \oint_{C_{air}+C_{water}} p\,n_\eta \cdot \zeta\, d\ell = \int_{C_{air}} p_0 n_\eta \cdot \zeta\, d\ell + \int_{C_{water}} (p_0+\gamma z)\, n_\eta \cdot \zeta\, d\ell \\ M_\zeta &= \oint_{C_{air}+C_{water}} p\,n_\zeta \cdot \eta\, d\ell = \int_{C_{air}} p_0 n_\zeta \cdot \eta\, d\ell + \int_{C_{water}} (p_0+\gamma z)\, n_\zeta \cdot \eta\, d\ell \end{aligned}\right\} \cdots\cdots\cdots (2.11)$$

Here, if the above moments expressed in terms of integrals with respect to the argument ϕ as in Eqs. (2.6) and (2.7) for $F_{-\eta}$ and $F_{-\zeta}$ in the previous section, M_η becomes as :

$$\begin{aligned} M_\eta &= \int_{\frac{\pi}{2}+\theta}^{\frac{3\pi}{2}+\theta} p_0 \sin\phi \cdot R\cos\phi \cdot Rd\phi + \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} (p_0+\gamma z)\sin\phi \cdot R\cos\phi \cdot Rd\phi \\ &= \frac{1}{2} p_0 R^2 \int_{-\frac{\pi}{2}+\theta}^{\frac{3\pi}{2}+\theta} \sin 2\phi\, d\phi + \gamma R^2 \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} z \sin\phi\cos\phi\, d\phi \\ &= \gamma R^2 \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} z(\phi)\sin\phi\cos\phi\, d\phi \end{aligned} \cdots\cdots\cdots (2.12)$$

And, M_ζ becomes as :

$$\begin{aligned} M_\zeta &= \int_{\frac{\pi}{2}+\theta}^{\frac{3\pi}{2}+\theta} p_0 \cos\phi \cdot R\sin\phi \cdot Rd\phi + \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} (p_0+\gamma z)\cos\phi \cdot R\sin\phi \cdot Rd\phi \\ &= \frac{1}{2} p_0 R^2 \int_{-\frac{\pi}{2}+\theta}^{\frac{3\pi}{2}+\theta} \sin 2\phi\, d\phi + \gamma R^2 \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} z \sin\phi\cos\phi\, d\phi \\ &= \gamma R^2 \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} z(\phi)\sin\phi\cos\phi\, d\phi \end{aligned} \cdots\cdots\cdots (2.13)$$

The above results show that both equations for M_η and M_ζ are equivalent. Thus, the total counterclockwise moment M_o around the origin o due to pressure is zero as follows :

$$M_o = M_\zeta - M_\eta = 0 \cdots\cdots\cdots (2.14)$$

This is confirmed by the fact that the pressure acts perpendicular to the cylinder surface, so it is all directed toward the center of the circle.

Then, in both Eqs. (2.12) and (2.13), the integration of $\sin 2\phi$ with respect to atmospheric pressure p_0 in the 1st term of 2nd line is zero. Hence, we can calculate only the 2nd term by using Eq. (2.3) for the water depth $z(\phi)$, as follows :

$$\begin{aligned} M_\eta &= M_\zeta \\ &= \gamma R^3 \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} (\cos\theta\cos\phi + \sin\theta\sin\phi)\sin\phi\cos\phi\, d\phi \\ &= \gamma R^3 \left\{ \cos\theta \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \sin\phi\cos^2\phi\, d\phi + \sin\theta \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \sin^2\phi\cos\phi\, d\phi \right\} \quad \cdots\cdots (2.15) \end{aligned}$$

So, if we put $p = \cos\phi$ for the 1st term and $q = \sin\phi$ for the 2nd term and do a substitution integral for each, we obtain by folding the integral interval in half, as follows :

$$\begin{aligned} M_\eta &= M_\zeta \\ &= 2\gamma R^3 \left(\cos\theta \int_0^{\sin\theta} p^2\, dp + \sin\theta \int_0^{\cos\theta} q^2\, dq \right) \\ &= 2\gamma R^3 \left(\cos\theta \cdot \frac{1}{3}\sin^3\theta + \sin\theta \cdot \frac{1}{3}\cos^3\theta \right) \\ &= \frac{2}{3}\gamma R^3 \sin\theta\cos\theta\,(\sin^2\theta + \cos^2\theta) = \frac{2}{3}\gamma R^3 \sin\theta\cos\theta \quad \cdots\cdots (2.16) \end{aligned}$$

2.4 Positioning of the center of pressure C_P for the semi-submerged circular cylinder

To locate the center of pressure C_P in $o-\eta\zeta$ coordinate system fixed to circular cylinder, the hydraulic method used in the authors' previous papers(34),(35),(46)~(54) is applied. This method was used by Ohgushi(9) for an example problem of the rolling gate.

Since the forces $F_{-\eta}$ and $F_{-\zeta}$ due to the hydrostatic pressure obtained in Section 2.1 act on the center of pressure $C_P\,(\eta_P, \zeta_P)$, the moments M_η and M_ζ due to the same pressure obtained in Section 2.3 can be expressed respectively, as follows :

$$\left.\begin{aligned} M_\eta &= F_{-\eta}\,\zeta_P \\ M_\zeta &= F_{-\zeta}\,\eta_P \end{aligned}\right\} \quad \cdots\cdots (2.17)$$

Therefore, the unknown coordinate (η_P, ζ_P) of the center of pressure C_P can be determined by Eq. (2.17). Here, the η-coordinate, η_P, can be calculated by using Eq. (2.9) for $F_{-\zeta}$ and the Eq. (2.16) for M_ζ due to the hydrostatic pressure in the $-\zeta$ direction, as follows :

$$\begin{aligned} \eta_P &= \frac{M_\zeta}{F_{-\zeta}} \\ &= \frac{\frac{2}{3}\gamma R^3 \sin\theta\cos\theta}{\gamma \frac{\pi R^2}{2}\cos\theta} = \frac{4}{3\pi} R\sin\theta \quad \cdots\cdots (2.18) \end{aligned}$$

Similarly, the ζ-coordinate, ζ_P, can be calculated by using Eq. (2.8) for $F_{-\eta}$ and Eq. (2.16) for M_η due to the hydrostatic pressure in the $-\eta$ direction, as follows :

$$\zeta_P = \frac{M_\eta}{F_{-\eta}} = \frac{\frac{2}{3}\gamma R^3 \sin\theta\cos\theta}{\gamma\frac{\pi R^2}{2}\sin\theta} = \frac{4}{3\pi}R\cos\theta \quad \cdots\cdots(2.19)$$

Let us consider the above equations. For ζ_P in Eq. (2.19), if we assume the upright state $\theta = 0$ from the beginning, $\sin\theta$ in the denominator $F_{-\eta}$ and numerator M_η will be zero, so the fraction becomes indeterminate forms and ζ_P cannot be determined. The reason is why we were able to locate the vertical component ζ_P of the center of pressure, the semi-submerged cylinder was laterally inclined along with its $\eta\zeta$-coordinate axes, even though the shape did not change when inclined.

On the other hand, for η_P in Eq. (2.18), even if the heel angle is $\theta = 0$ from the beginning, the denominator $F_{-\zeta}$ can take a finite value because of $\cos\theta = 1$, and horizontal component η_P can be determined.

From the results of both equations above, the coordinates (η_P, ζ_P) of the center of pressure C_P are determined as :

$$(\eta_P, \zeta_P) = \left(\frac{4}{3\pi}R\sin\theta, \frac{4}{3\pi}R\cos\theta\right) \quad \cdots\cdots(2.20)$$

The above (η_P, ζ_P) coordinates fixed to the inclined cylinder are transformed to (y_P, z_P) coordinates fixed to space, as follows :

$$\left.\begin{aligned} y_P &= \eta_P\cos\theta - \zeta_P\sin\theta \\ &= \frac{4}{3\pi}R(\sin\theta\cdot\cos\theta - \cos\theta\cdot\sin\theta) = 0 \\ z_P &= \zeta_P\cos\theta + \eta_P\sin\theta \\ &= \frac{4}{3\pi}R(\cos^2\theta + \sin^2\theta) = \frac{4}{3\pi}R \end{aligned}\right\} \quad \cdots\cdots(2.21)$$

Therefore, the center of pressure C_P in the space-fixed coordinate is located as :

$$(y_P, z_P) = \left(0, \frac{4}{3\pi}R\right) \quad \cdots\cdots(2.22)$$

This correctly indicates the figure centroid on the centerline (*i.e.* z-axis) of the semicircle below the water surface. Hence, it is proved that the center of hydrostatic pressure is equal to the well-known center of buoyancy, even for the shape of a semi-submerged circular cylinder.

2.5 Considerations

In the case of the semi-submerged circular cylinder in this chapter, the situation differs from that of a rectangle or an arbitrary cross-sectional shape in authors' previous papers [(34),(35),(49)~(54)] and of a triangular prism in the next chapter. The reason is why its geometrical shape under the water surface does not change even when the circular cylinder is inclined laterally. As a result, it is not necessary to determine the center of pressure in the upright position by setting the lateral inclination angle θ to zero. So, its position can be computed by coordinate transformation, as shown in Eq. (2.21) of the previous section.

Therefore, it was also found that the center of pressure can be positioned by tilting the coordinate system in a way that it is shifted from the vertical direction, without inclining the floating body as advocated by Yabushita *et al.* [(42)].

3. Positioning of the Center of Hydrostatic Pressure C_P Acting on the Triangular Prism

Fig. 3.1 shows that a cross-section of triangular Prism (breadth $2b$ draft f, freeboard h, vertex angle 2ϕ) inclines laterally with a heel angle θ to the starboard side. Here, the half breadth b of the waterline of the triangular prism in the upright state can be written, using the draft f and the half vertex angle ϕ, as follows :

$$b = f\tan\phi \quad \cdots\cdots (3.1)$$

Here, the cross-section of this triangular prism is an isosceles triangle with base (*i.e.* deck length) $2(f+h)\tan\phi$, height $f+h$, and both sides $(f+h)\sec\phi$.

3.1 Preparation calculations, including wetted lengths on both port and starboard sides

Let's consider the exposed triangle (port side, L for short) ΔoE_LT_L and the immersed triangle (starboard side, R for short) ΔoE_RT_R near the waterline in Fig. 3. 1. The heights $q_L = \overline{U_LT_L}$ and $q_R = \overline{U_RT_R}$ of each triangle can be expressed geometrically in two ways, using $x_L = \overline{U_LE_L}$ and $x_R = \overline{U_RE_R}$, as follows :

$$\left.\begin{aligned} q_L &= (b - x_L)\tan\theta = \frac{x_L}{\tan\phi} \\ q_R &= (b + x_R)\tan\theta = \frac{x_R}{\tan\phi} \end{aligned}\right\} \quad \cdots\cdots (3.2)$$

Thus, for x_L and x_R, the following relations can be obtained respectively as :

$$\left.\begin{aligned} x_L &= (b - x_L)\tan\phi\tan\theta \\ x_R &= (b + x_R)\tan\phi\tan\theta \end{aligned}\right\} \quad \cdots\cdots (3.3)$$

Therefore, x_L and x_R can be solved by using the relation in Eq. (3.1) for the half breadth b as follows :

$$\left.\begin{aligned} x_L &= \frac{\varepsilon}{1+\varepsilon} f \tan\phi \\ x_R &= \frac{\varepsilon}{1-\varepsilon} f \tan\phi \end{aligned}\right\} \quad \cdots\cdots (3.4)$$

Here, ε in the above equation is defined as the product of the tangent of the half vertex angle ϕ and that of the heel angle θ, as follows:

$$\varepsilon \equiv \tan\phi \tan\theta \quad \cdots\cdots (3.5)$$

Next, the decremental length s_L and the incremental length s_R of the wetted length on the port and starboard sides respectively, are written as :

$$\left.\begin{aligned} s_L &= \frac{x_L}{\sin\phi} = \frac{\varepsilon}{1+\varepsilon} f \sec\phi \\ s_R &= \frac{x_R}{\sin\phi} = \frac{\varepsilon}{1-\varepsilon} f \sec\phi \end{aligned}\right\} \quad \cdots\cdots (3.6)$$

Therefore, the wetted lengths ℓ_L and ℓ_R on the port and starboard sides are obtained as follows :

$$\left.\begin{aligned} \ell_L &= f \sec\phi - s_L = \frac{1}{1+\varepsilon} f \sec\phi \\ \ell_R &= f \sec\phi + s_R = \frac{1}{1-\varepsilon} f \sec\phi \end{aligned}\right\} \quad \cdots\cdots (3.7)$$

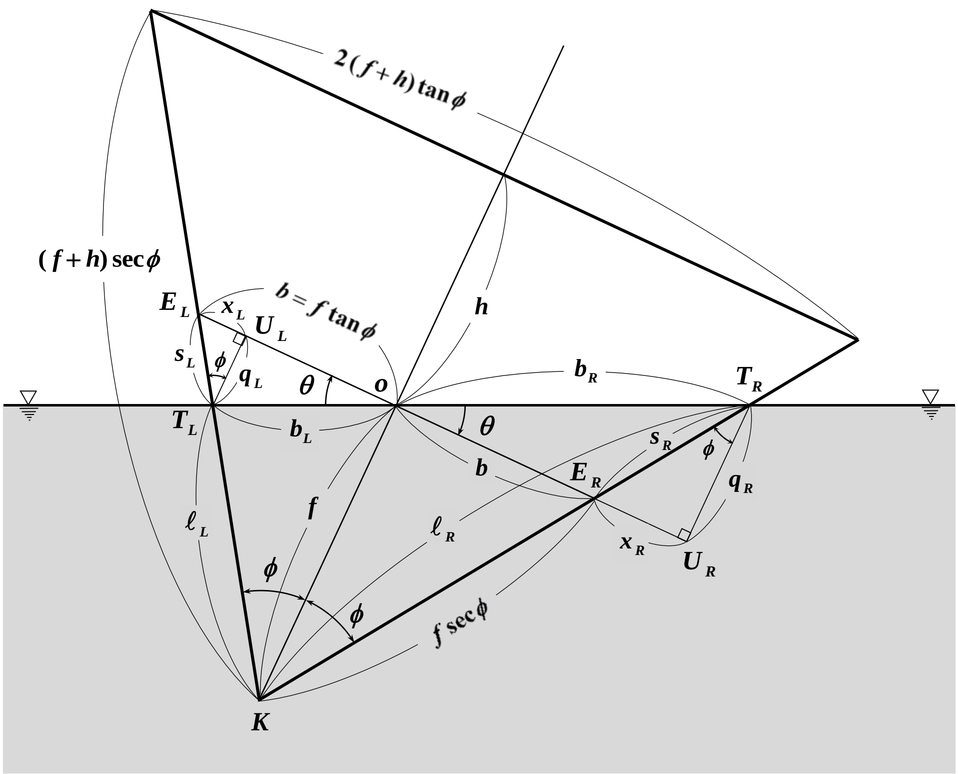

Fig. 3.1 Cross-section of an inclined triangular prism.

The waterline widths b_L and b_R on both the port and starboard sides can be obtained by using x_L and x_R in Eq. (3.3) as follows :

$$\left.\begin{aligned} b_L &= (b-x_L)\sec\theta = \frac{1}{1+\varepsilon} f\tan\phi\sec\theta \\ b_R &= (b+x_R)\sec\theta = \frac{1}{1-\varepsilon} f\tan\phi\sec\theta \end{aligned}\right\} \qquad (3.8)$$

Thus, the total waterline breadth is written as :

$$b_L + b_R = \frac{2}{1-\varepsilon^2} f\tan\phi\sec\theta \qquad (3.9)$$

Therefore, the area A of triangle $\triangle K T_L T_R$ below the water surface of a triangular prism, which is inclined laterally with heel angle θ, is obtained as follows :

$$A = \frac{1}{2}(b_L + b_R)\cdot f\cos\theta = \frac{1}{1-\varepsilon^2} f^2\tan\phi \qquad (3.10)$$

Since the underwater area A_0 in the upright state ($\theta=0$ *i.e.* $\varepsilon=0$) is shown below, the underwater area A in the above inclined state is increased by $\frac{\varepsilon^2}{1-\varepsilon^2}A_0$ from the upright state.

$$A_0 \equiv A\big]_{\theta=0} = f^2\tan\phi \qquad (3.11)$$

3.2 Forces due to hydrostatic pressure acting on three surfaces around a triangular prism

Fig. 3.2 shows the pressure distribution and the forces generated by integrating it, acting on the cross-section of the triangular prism drawn in Fig. 3.1. The coordinate systems are $o-yz$ fixed in space with the z-axis pointing vertically downward, and $o-\eta\zeta$ fixed on the prism and tilted, both with the origin o at the center of still water surface.

The atmospheric pressure is denoted by p_0 and the specific gravity of water is denoted by γ. The atmospheric pressure p_0 is shown as a dashed line, and the hydrostatic pressure γz as a solid line. The respective pressures are shown as thin vectors, and the forces as thick vectors. Then, all are acting perpendicularly to the surface of the triangular prism.

The depth Z_f at the vertex K of the triangle corresponding to the ship's bottom is denoted as :

$$Z_f = f\cos\theta \qquad (3.12)$$

The forces P_{Left} and P_{Right} acting on the port (subscripts in Left) and starboard (subscripts in Right) sides are obtained by summing the forces $P_{Left}^{(0)}$, $P_{Right}^{(0)}$ due to uniformly distributed atmospheric pressure acting on the entire port side and the forces $P_{Left}^{(0)}$, $P_{Right}^{(0)}$ due to the triangularly distributed hydrostatic pressure acting on the submerged part respectively, by using the wetted lengths ℓ_L , ℓ_R in Eq. (3.7), as follows :

$$\left.\begin{aligned}
P_{Left} &= P_{Left}^{(0)} + P_{Left}^{(\gamma)} \\
&= p_0(f+h)\sec\phi + \frac{1}{2}\gamma Z_f \ell_L \\
&= p_0(f+h)\sec\phi + \frac{1}{2}\gamma f^2 \frac{\sec\phi\cos\theta}{1+\varepsilon} \\
P_{Right} &= P_{Right}^{(0)} + P_{Right}^{(\gamma)} \\
&= p_0(f+h)\sec\phi + \frac{1}{2}\gamma Z_f \ell_R \\
&= p_0(f+h)\sec\phi + \frac{1}{2}\gamma f^2 \frac{\sec\phi\cos\theta}{1-\varepsilon}
\end{aligned}\right\} \quad \cdots\cdots (3.13)$$

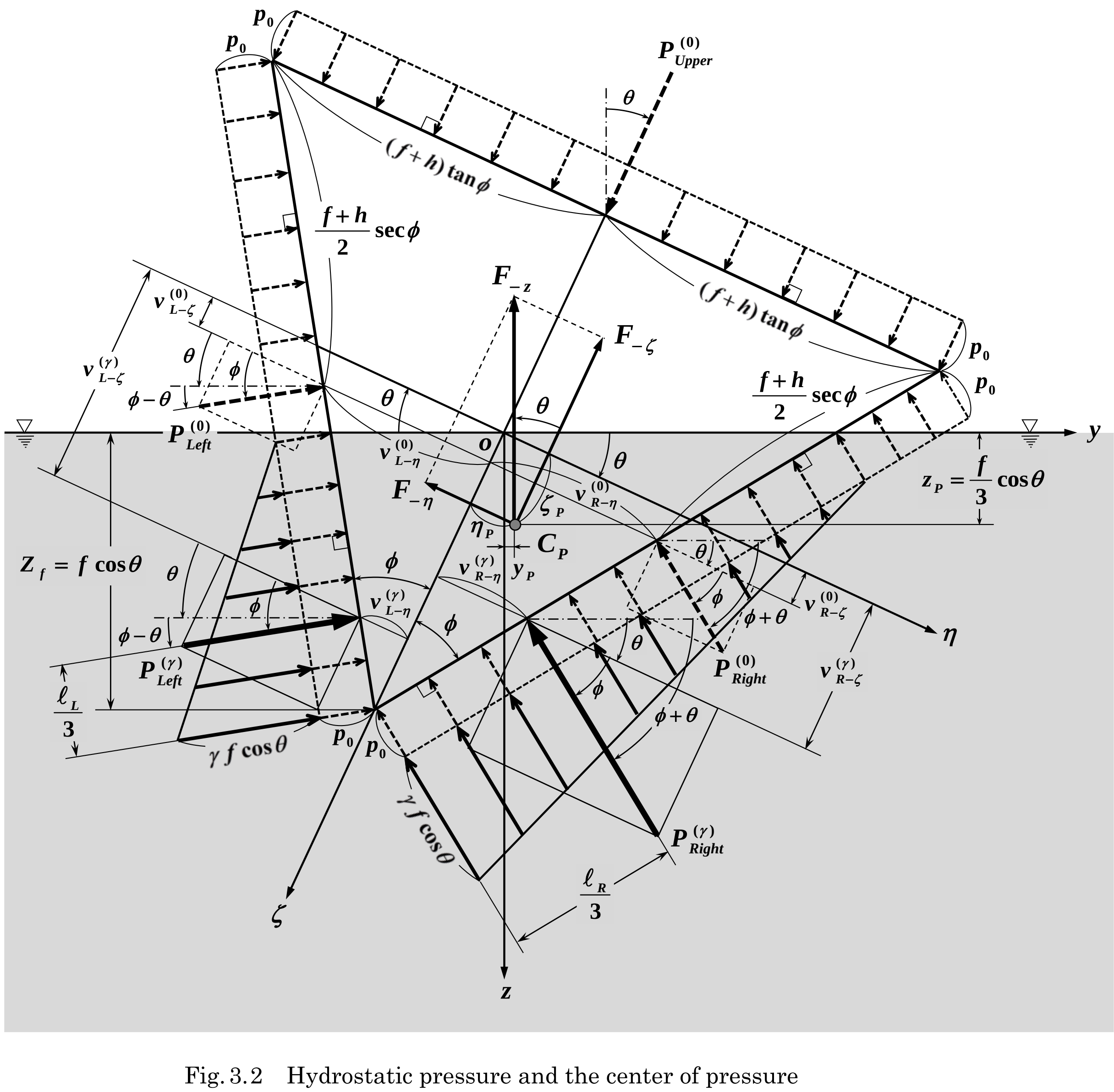

Fig. 3.2 Hydrostatic pressure and the center of pressure acting on the cross-section of an inclined triangular prism.

The force P_{Upper} acting on the deck (subscripts in Upper) is only $P_{Upper}^{(0)}$ due to atmospheric pressure, so it is obtained as :

$$\begin{aligned} P_{Upper} &= P_{Upper}^{(0)} \\ &= 2p_0(f+h)\tan\phi \end{aligned} \qquad \cdots\cdots (3.14)$$

3.3 Combined forces $F_{-\eta}$ and $F_{-\zeta}$ in the $-\eta$ and $-\zeta$ directions acting on the prism surface

The combined forces $F_{-\eta}$ and $F_{-\zeta}$ acting in the $-\eta$ and $-\zeta$ directions fixed to the inclined floating prism are obtained by using P_{Left} , P_{Right} and P_{Upper} in Eqs. (3.13) and (3.14), as follows :

$$\left.\begin{aligned} F_{-\eta} &= P_{Right}\cos\phi - P_{Left}\cos\phi \\ &= \frac{1}{2}\gamma f^2\cos\theta\left(\frac{1}{1-\varepsilon}-\frac{1}{1+\varepsilon}\right) + p_0(f+h) - p_0(f+h) \\ &= \gamma f^2\frac{\varepsilon}{1-\varepsilon^2}\cos\theta = \gamma A\sin\theta \\ F_{-\zeta} &= P_{Right}\sin\phi + P_{Left}\sin\phi - P_{Upper} \\ &= \frac{1}{2}\gamma f^2\tan\phi\cos\theta\left(\frac{1}{1-\varepsilon}+\frac{1}{1+\varepsilon}\right) + 2p_0(f+h)\tan\phi - P_{Upper}^{(0)} \\ &= \gamma f^2\frac{\tan\phi}{1-\varepsilon^2}\cos\theta = \gamma A\cos\theta \end{aligned}\right\} \qquad \cdots\cdots (3.15)$$

Here, $F_{-\eta}$ and $F_{-\zeta}$ of the above are obtained as the sine and cosine components of the buoyant force γA, as shown by F_{-z} of Eq. (3.17) in the next section, with respect to the heel angle θ. This result indicates that the atmospheric pressure p_0 cancels out and does not contribute to the combined forces acting on the floating prism.

3.4 Forces F_{-y} and F_{-z} converted in the $-y$ and $-z$ directions

The horizontal component (in the $-y$ direction) F_{-y} and the vertical component (in the $-z$ direction) F_{-z} are obtained by coordinate transformation of $F_{-\eta}$ and $F_{-\zeta}$ in Eq. (3.15) of the previous section.

Then, the horizontal component F_{-y} is transformed as :

$$\begin{aligned} F_{-y} &= F_{-\eta}\cos\theta - F_{-\zeta}\sin\theta \\ &= \gamma A(\sin\theta\cos\theta - \cos\theta\sin\theta) \\ &= 0 \end{aligned} \qquad \cdots\cdots (3.16)$$

From the above result, the horizontal component of the combined force does not generate even in an left-right asymmetric pressure field due to lateral inclination.

And, the vertical component F_{-z} is similarly transformed as :

$$
\begin{aligned}
F_{-z} &= F_{-\zeta}\cos\theta + F_{-\eta}\sin\theta \\
&= \gamma A(\cos^2\theta + \sin^2\theta) \\
&= \gamma A \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (3.17)
\end{aligned}
$$

The above result shows that the vertical component is obtained by the product of the specific gravity γ of water and the cross-sectional area A under the water surface of the triangular prism shown in Eq. (3.10). This indicates that F_{-z} is the very buoyant force taught by Archimedes' principle(1).

On the other hand, the F_{-y} and F_{-z} can also be obtained directly from P_{Left} , P_{Right} and P_{Upper} in Eqs. (3.13) and (3.14), as follows :

First, the horizontal component F_{-y} is calculated as :

$$
\begin{aligned}
F_{-y} &= P_{Right}\cos(\phi+\theta) - P_{Left}\cos(\phi-\theta) + P_{Upper}\sin\theta \\
&= \frac{1}{2}\gamma f^2 \sec\phi\cos\theta\left\{\frac{\cos(\phi+\theta)}{1-\varepsilon} - \frac{\cos(\phi-\theta)}{1+\varepsilon}\right\} \\
&\quad + p_0(f+h)\left[\sec\phi\left\{\cos(\phi+\theta)-\cos(\phi-\theta)\right\} + 2\tan\phi\sin\theta\right] \\
&= -\gamma f^2\sec\phi\cos\theta\ \frac{\sin\phi\sin\theta - \varepsilon\cos\phi\cos\theta}{1-\varepsilon^2} - 2p_0(f+h)\sin\theta(\sin\phi\sec\phi - \tan\phi) \\
&= 0 \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (3.18)
\end{aligned}
$$

Next, the vertical component F_{-z} is calculated as :

$$
\begin{aligned}
F_{-z} &= P_{Right}\sin(\phi+\theta) + P_{Left}\sin(\phi-\theta) - P_{Upper}\cos\theta \\
&= \frac{1}{2}\gamma f^2\sec\phi\cos\theta\left\{\frac{\sin(\phi+\theta)}{1-\varepsilon} + \frac{\sin(\phi-\theta)}{1+\varepsilon}\right\} \\
&\quad + p_0(f+h)\left[\sec\phi\left\{\sin(\phi+\theta)+\sin(\phi-\theta)\right\} - 2\tan\phi\cos\theta\right] \\
&= \gamma f^2\sec\phi\cos\theta\ \frac{\sin\phi\cos\theta + \varepsilon\cos\phi\sin\theta}{1-\varepsilon^2} + 2p_0(f+h)\cos\theta(\sin\phi\sec\phi - \tan\phi) \\
&= \gamma f^2\sec\phi\cos\theta\ \frac{\sin\phi\sec\theta}{1-\varepsilon^2} = \gamma f^2\frac{1}{1-\varepsilon^2}\tan\phi \\
&= \gamma A \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (3.19)
\end{aligned}
$$

Both of the above equations cancel out the atmospheric pressure p_0 and are identical to Eqs. (3.16) and (3.17) obtained by transforming the coordinates of $F_{-\eta}$ and $F_{-\eta}$. This confirms that the forces due to pressure in Section 3.2 have been calculated correctly.

3.5 Moments M_η and M_ζ due to pressure in the η and ζ directions acting on the prism surface

Consider the calculation of the moment M_η about the origin o , generated by the η - directional components of the forces P_{Left} and P_{Right} due to pressure acting perpendicularly on the sides of a triangular prism.

The levers $v_{L-\zeta}^{(0)}$ and $v_{R-\zeta}^{(0)}$ parallel to the η - axis on both the port and starboard sides by the atmospheric pressure components $^{(0)}$ of the uniform distribution are obtained as the same length on both sides, since the both side lengths including freeboard are $(f+h)\sec\phi$, as follows :

$$v_{L-\zeta}^{(0)} = v_{R-\zeta}^{(0)} = f - \frac{(f+h)\sec\phi}{2}\cos\phi = \frac{f-h}{2} \quad \cdots\cdots (3.20)$$

The levers $v_{L-\zeta}^{(\gamma)}$ and $v_{R-\zeta}^{(\gamma)}$ parallel to the ζ - axis on both port and starboard sides due to the hydrostatic components $^{(\gamma)}$ of the triangular distribution are obtained by using the wetted lengths ℓ_L and ℓ_R in Eq. (3.7) as follows :

$$\left.\begin{aligned} v_{L-\zeta}^{(\gamma)} &= f - \frac{\ell_L}{3}\cos\phi = f - \frac{1}{3(1+\varepsilon)} f = \frac{2+3\varepsilon}{3(1+\varepsilon)} f \\ v_{R-\zeta}^{(\gamma)} &= f - \frac{\ell_R}{3}\cos\phi = f - \frac{1}{3(1-\varepsilon)} f = \frac{2-3\varepsilon}{3(1-\varepsilon)} f \end{aligned}\right\} \quad \cdots\cdots (3.21)$$

By the above two equations, the clockwise moment M_η due to pressure in the η - direction about the origin o can be obtained independently of the atmospheric pressure p_0, using Eqs. (3.13), (3.20) and (3.21), as follows :

$$\begin{aligned} M_\eta =\ & P_{Right}^{(0)}\cos\phi\cdot v_{R-\zeta}^{(0)} + P_{Right}^{(\gamma)}\cos\phi\cdot v_{R-\zeta}^{(\gamma)} \\ & - (P_{Left}^{(0)}\cos\phi\cdot v_{L-\zeta}^{(0)} + P_{Left}^{(\gamma)}\cos\phi\cdot v_{L-\zeta}^{(\gamma)}) \\ =\ & \frac{1}{6}\gamma f^3\cos\theta\left\{\frac{2-3\varepsilon}{(1-\varepsilon)^2} - \frac{2+3\varepsilon}{(1+\varepsilon)^2}\right\} + p_0(f+h)\cdot(v_{R-\zeta}^{(0)} - v_{L-\zeta}^{(0)}) \\ =\ & \frac{1}{3}\gamma f^3\frac{\varepsilon(1-3\varepsilon^2)}{(1-\varepsilon^2)^2}\cos\theta = \frac{1}{3}\gamma f A\frac{1-3\varepsilon^2}{1-\varepsilon^2}\sin\theta \quad \cdots\cdots (3.22) \end{aligned}$$

Next, the moment M_ζ around point o, generated by P_{Upper} and the ζ - directional components of P_{Left} and P_{Right}, is calculated.

The levers $v_{L-\eta}^{(0)}$ and $v_{R-\eta}^{(0)}$ parallel to the η - axis due to the atmospheric pressure components $^{(0)}$ are obtained as :

$$v_{L-\eta}^{(0)} = v_{R-\eta}^{(0)} = \frac{(f+h)\sec\phi}{2}\sin\phi = \frac{f+h}{2}\tan\phi \quad \cdots\cdots (3.23)$$

Here, the above equation, like Eq. (3.20), has the same length on both sides.

The levers $v_{L-\eta}^{(\gamma)}$ and $v_{L-\eta}^{(\gamma)}$ parallel to the η - axis on both the port and starboard sides due to the hydrostatic components $^{(\gamma)}$ can be obtained by using ℓ_L and ℓ_R in Eq. (3.7), as follows :

$$\left.\begin{aligned} v_{L-\eta}^{(\gamma)} &= \frac{\ell_L}{3}\sin\phi = \frac{\tan\phi}{3(1+\varepsilon)} f \\ v_{R-\eta}^{(\gamma)} &= \frac{\ell_R}{3}\sin\phi = \frac{\tan\phi}{3(1-\varepsilon)} f \end{aligned}\right\} \quad \cdots\cdots (3.24)$$

Therefore, the counterclockwise moment M_ζ due to pressure in the ζ - direction about the origin o can be calculated by Eqs. (3.13), (3.23) and (3.24), as follows :

$$
\begin{aligned}
M_\zeta &= P^{(0)}_{Right}\sin\phi\cdot v^{(0)}_{R-\eta} + P^{(\gamma)}_{Right}\sin\phi\cdot v^{(\gamma)}_{R-\eta} \\
&\quad -(P^{(0)}_{Left}\sin\phi\cdot v^{(0)}_{L-\eta} + P^{(\gamma)}_{Left}\sin\phi\cdot v^{(\gamma)}_{L-\eta}) + P_{Upper}\times 0 \\
&= \frac{1}{6}\gamma f^3\tan^2\phi\cos\theta\left\{\frac{1}{(1-\varepsilon)^2}-\frac{1}{(1+\varepsilon)^2}\right\} + p_0(f+h)\tan\phi\cdot(v^{(0)}_{R-\eta}-v^{(0)}_{L-\eta}) \\
&= \frac{2}{3}\gamma f^3\frac{\varepsilon\tan^2\phi}{(1-\varepsilon^2)^2}\cos\theta = \frac{2}{3}\gamma f A\frac{\tan^2\phi}{1-\varepsilon^2}\sin\theta \qquad \cdots\cdots (3.25)
\end{aligned}
$$

Here, M_ζ, like M_η, is obtained independently of the atmospheric pressure p_0.

3.6 Positioning of the center of pressure C_P for the triangular prism at lateral inclination

The center of pressure C_P is located in $o-\eta\zeta$ coordinate system fixed to the inclined triangular prism, as in the case of the semi-submerged circular cylinder in Chapter 2. According to the hydraulic method used by Ohgushi[(9)], the moments and forces due to pressure are related by Eq. (2.17), assuming the coordinates (η_P, ζ_P) of center of pressure.

Therefore, the η - coordinate η_P can be determined by the combined force $F_{-\zeta}$ and moment M_ζ due to pressure in the $-\zeta$ - direction, by using Eq. (3.25) and the latter in Eq. (3.15), as follows :

$$
\begin{aligned}
\eta_P &= \frac{M_\zeta}{F_{-\zeta}} = \frac{\frac{2}{3}\gamma f A\frac{\tan^2\phi}{1-\varepsilon^2}\sin\theta}{\gamma A\cos\theta} \\
&= \frac{2}{3}f\frac{\varepsilon\tan\phi}{1-\varepsilon^2} \qquad \cdots\cdots (3.26)
\end{aligned}
$$

And, the ζ - coordinate ζ_P can be determined by the combined force $F_{-\eta}$ and moment M_η due to pressure in the $-\eta$ - direction, by using Eq. (3.22) and the former in Eq. (3.15), as follows :

$$
\begin{aligned}
\zeta_P &= \frac{M_\eta}{F_{-\eta}} = \frac{\frac{1}{3}\gamma f A\frac{1-3\varepsilon^2}{1-\varepsilon^2}\sin\theta}{\gamma A\sin\theta} \\
&= \frac{1}{3}f\frac{1-3\varepsilon^2}{1-\varepsilon^2} \qquad \cdots\cdots (3.27)
\end{aligned}
$$

Considering the above, ζ_P of vertical component can be obtained by offsetting the zero factor $\sin\theta$ at the heel angle $\theta\to 0$ with the denominator and numerator, as shown in Eq. (3.27). Here, if we start the calculation from the beginning as the upright state with $\theta = 0$, both the denominator $F_{-\eta}$ and the numerator M_η are in equilibrium and become zero, so the fraction becomes indeterminate forms and ζ_P cannot be determined. This is the reason why we were able to determine the position of the center of pressure in the ζ - direction by inclining the floating body laterally.

On the other hand, in the calculation of η_P in Eq. (3.26), even if the heel angle is $\theta = 0$ from the beginning, the numerator M_ζ is in equilibrium and zero, but the denominator $F_{-\zeta}$ takes a finite value as the cosine component of the buoyancy. Therefore, the horizontal component η_P can be determine, even if we start the calculation as the upright state.

Let us now transform the resulting center of pressure $C_P(\eta_P, \zeta_P)$ in the floating prism-fixed coordinates into the space-fixed coordinate system (y_P, z_P).

First, y_P in the horizontal direction becomes as :

$$\begin{aligned} y_P &= \eta_P \cos\theta - \zeta_P \sin\theta \\ &= \frac{1}{3} f \frac{2\varepsilon \tan\phi \cos\theta - (1-3\varepsilon^2)\sin\theta}{1-\varepsilon^2} \\ &= \frac{1}{3} f \frac{2\tan^2\phi + 3\varepsilon^2 - 1}{1-\varepsilon^2} \sin\theta \qquad \cdots\cdots (3.28) \end{aligned}$$

Next, z_P in the vertical direction becomes as :

$$\begin{aligned} z_P &= \zeta_P \cos\theta + \eta_P \sin\theta \\ &= \frac{1}{3} f \frac{(1-3\varepsilon^2)\cos\theta + 2\varepsilon \tan\phi \sin\theta}{1-\varepsilon^2} \\ &= \frac{1}{3} f \frac{(1-3\varepsilon^2) + 2\varepsilon^2}{1-\varepsilon^2} \cos\theta = \frac{1}{3} f \cos\theta \qquad \cdots\cdots (3.29) \end{aligned}$$

From the above results, it is clear that the latter z_P indicates the vertical position of figure centroid of a triangle of height $f\cos\theta$, with the water surface as its base. Hence, we will verify in the next section whether the former y_P also coincides with the horizontal position of figure centroid of underwater triangle.

3.7 Verification by the position of the figure centroid of the triangle below the water surface

Fig. 3.3 shows an extract of the area under the water surface for the cross-section of the triangular prism in Fig. 3.2. Let us divide the triangle $\triangle K T_L T_R$ into two parts by the z' - axis connecting the vertex K of the triangle and the origin o' taken vertically above the vertex K.

For the Left triangle $\triangle K o' T_L$, the area is A_L and the base is y_L, and for the Right triangle $\triangle K o' T_R$, the area is A_R and the base is y_R. And the height is the common on both left and right triangles, $\overline{o'K} = Z_f$.

In this case, the areas A_L and A_R of the left and right triangles respectively, are written as :

$$\left.\begin{aligned} A_L &= \frac{1}{2} Z_f \, y_L \\ A_R &= \frac{1}{2} Z_f \, y_R \end{aligned}\right\} \qquad \cdots\cdots (3.30)$$

The base of the triangle $\triangle K T_L T_R$ can be written in the following two ways, by using y_L and y_R in Fig. 3.3 and b_L and b_R in Fig. 3.1.

$$y_L + y_R = b_L + b_R \;\; (= \mathit{base\ of}\ \triangle K T_L T_R) \qquad \cdots\cdots (3.31)$$

Therefore, the area A of $\triangle K T_L T_R$, which is the sum of A_L and A_R above, is expressed by Eq. (3.10) in Section 3.1, as follows :

$$\begin{aligned} A &= A_L + A_R \\ &= \frac{1}{2} Z_f \, (y_L + y_R) \\ &= \frac{1}{2} f \cos\theta \, (b_L + b_R) = \frac{1}{1-\varepsilon^2} f^2 \tan\phi \qquad \cdots\cdots (3.32) \end{aligned}$$

And, y_L and y_R, which correspond to the bases of the two halves of $\triangle K T_L T_R$, become respectively, using ϕ and θ, as follows :

$$\left.\begin{aligned} y_L &= Z_f \tan(\phi-\theta) = Z_f \frac{\tan\phi - \tan\theta}{1+\varepsilon} \\ y_R &= Z_f \tan(\phi+\theta) = Z_f \frac{\tan\phi + \tan\theta}{1-\varepsilon} \end{aligned}\right\} \qquad \cdots\cdots (3.33)$$

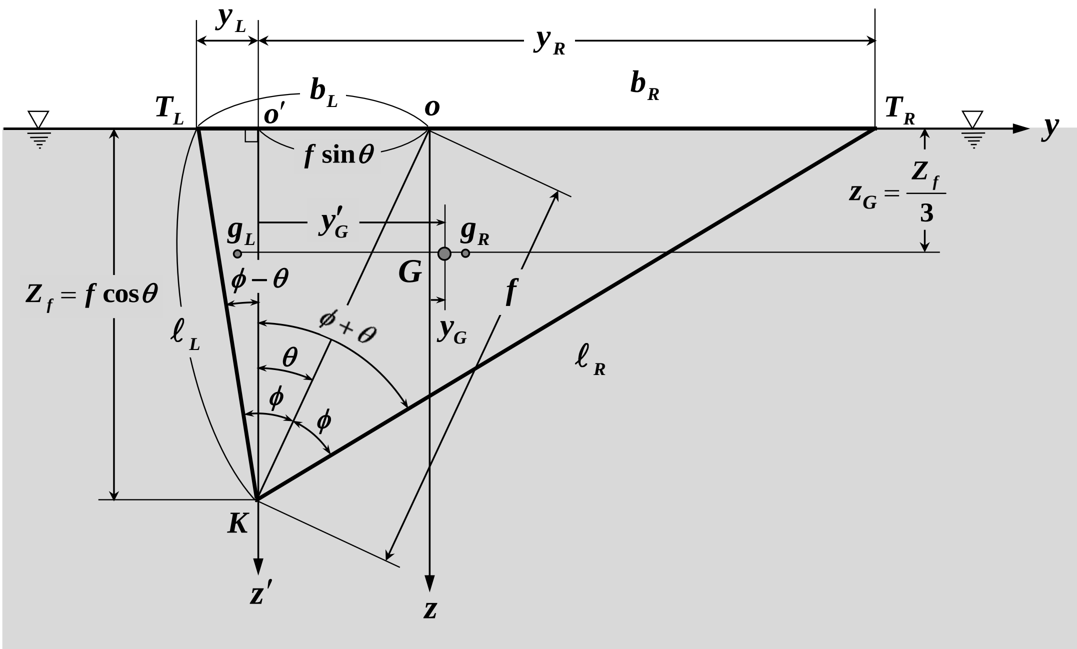

Fig. 3.3 Figure centroid of triangular cross-section below the water surface.

The areal moment $M_z{}'$ of triangle $\Delta K T_L T_R$ about the z'-axis can be obtained by using Eq. (3.30) for A_L and A_R as follows, since the horizontal distance from the z'-axis to the figure centroids g_L and g_R of the divided left and right triangles $\Delta K o' T_L$ and $\Delta K o' T_R$ respectively, is the lever of moment.

$$\begin{aligned} M_z{}' &= A_R \times \frac{y_R}{3} - A_L \times \frac{y_L}{3} \\ &= \frac{1}{6} Z_f \,(y_R{}^2 - y_L{}^2) \end{aligned} \qquad \cdots\cdots (3.34)$$

Proceeding with the calculation, by using Eq. (3.33) for y_L and y_R, Eq. (3.12) for Z_f, and Eq. (3.5) for ε, the moment $M_z{}'$ can be obtained in terms of A in Eq. (3.32), as follows :

$$\begin{aligned} M_z{}' &= \frac{1}{6} Z_f{}^3 \left\{ \frac{(\tan\phi + \tan\theta)^2}{(1-\varepsilon)^2} - \frac{(\tan\phi - \tan\theta)^2}{(1+\varepsilon)^2} \right\} \\ &= \frac{1}{6} Z_f{}^3 \, \frac{4\,\varepsilon \sec^2\phi \sec^2\theta}{(1-\varepsilon^2)^2} \\ &= \frac{2}{3} f^{\,3} \, \frac{\tan\phi \sec^2\phi}{(1-\varepsilon^2)^2} \sin\theta = \frac{2}{3} f A \, \frac{\sec^2\phi}{1-\varepsilon^2} \sin\theta \end{aligned} \qquad \cdots\cdots (3.35)$$

Therefore, the horizontal distance $y_G{}'$ of the figure centroid G of triangular $\Delta K T_L T_R$ from the z'-axis is determined by dividing $M_z{}'$ in Eq. (3.35) by the area A in Eq. (3.32), as follows :

$$\begin{aligned} y_G{}' &= \frac{M_z{}'}{A} \\ &= \frac{2}{3} f \, \frac{\sec^2\phi}{1-\varepsilon^2} \sin\theta \end{aligned} \qquad \cdots\cdots (3.36)$$

Finally, consider finding the horizontal distance y_G of the figure centroid G from the original z-axis. Here, the distance $\overline{o'o}$ between the two origin points becomes as follows, by using Fig. 3.3 or the former part of Eqs. (3.8) and (3.33).

$$\overline{o'o} = b_L - y_L = f \sin\theta \qquad \cdots\cdots (3.37)$$

Hence, y_G is calculated by using Eqs. (3.36) and (3.37), as follows :

$$\begin{aligned} y_G &= y_G{}' - \overline{o'o} \\ &= y_G{}' - f \sin\theta \\ &= \frac{1}{3} f \, \frac{2\sec^2\phi - 3\,(1-\varepsilon^2)}{1-\varepsilon^2} \sin\theta = \frac{1}{3} f \, \frac{2\tan^2\phi + 3\varepsilon^2 - 1}{1-\varepsilon^2} \sin\theta \end{aligned} \qquad \cdots\cdots (3.38)$$

On the other hand, the vertical distance z_G from the y-axis to the figure centroid G, need not be calculated, since $\Delta K T_L T_R$ is a triangle of height $Z_f = f \cos\theta$ whose base is the water surface (*i.e.* y-axis), and is obtained as :

$$z_G = \frac{1}{3} f \cos\theta \qquad \cdots\cdots (3.39)$$

Thus, by comparing Eqs. (3.28) and (3.38) and Eqs. (3.29) and (3.39), we find as follows :

$$\left.\begin{array}{l} y_P = y_G \\ z_P = z_G \end{array}\right\} \qquad \cdots\cdots (3.40)$$

This result proves that the center of hydrostatic pressure is the well-known position of the center of buoyancy, since it indicates that the center of pressure of the asymmetrical triangular cross - section at lateral inclination coincides with the figure centroid below the water surface.

3.8 Positioning of the center of pressure C_P for the upright triangular prism

In order to clarify the consequences obtained in Eq. (3.40) of the previous section, we find the position of the center of pressure C_P of the triangular prism in the upright state. As a final step, let us set $\theta \to 0$ in the coordinates (η_P, ζ_P) of the center of pressure obtained for the inclined state.

Here, if the heel angle θ tends to zero, ε in Eq. (3.5) becomes as :

$$\varepsilon\,\big]_{\theta=0} = \tan\phi \tan\theta\,\big]_{\theta=0} = 0 \qquad \cdots\cdots (3.41)$$

Thus, by Eqs. (3.26) and (3.27) in Section 3.6, C_P is determined as follows :

$$C_P\,(\eta_P, \zeta_P)\,\big]_{\theta=0} = \left(0, \frac{f}{3}\right) \qquad \cdots\cdots (3.42)$$

Alternatively, since the $o-\eta\zeta$ and $o-yz$ coordinate systems coincide in the case of $\theta \to 0$, the following conclusion can be obtained by Eqs. (3.28) and (3.29) as well.

$$C_P\,(y_P, z_P)\,\big]_{\theta=0} = \left(0, \frac{f}{3}\right) \qquad \cdots\cdots (3.43)$$

Since the both of Eqs. (3.42) and (3.43) above clearly show the position of the figure centroid of the isosceles triangle below the water surface, it is proved that the center of hydrostatic pressure is the well-known center of buoyancy for the triangular prism.

4. Concluding Remarks

In this 2nd report, we proved that the center of hydrostatic pressure is equal to the well-known center of buoyancy (*i.e.* the figure centroid of the underwater area) for the typical cross-sectional shapes of semi-submerged circular cylinder and triangular prism, as in the case of the rectangular shape [1st half of (53)] reported earlier.

Although these two shapes are included in the proof of arbitrary shapes [2nd half of (53)], there is an essential difference between the two proofs. The reason is why the former does not change its underwater shape when inclined laterally, while the latter, like the rectangle, changes its cross-sectional shape when inclined. The present paper provided clear proofs for both shapes.

Acknowledgments

In closing this paper, I would like to express my heartfelt gratitude to *Dr. Yoshihiro KOBAYASHI*, former professor of *Sojo University* and current president of *Como-Techno Co., Ltd.* in Nagasaki, Japan. He always gave warm encouragement to the author's research and recommended that this study should be published in English. We are greatly inspired by the vigorous academic spirit with which he writes about the results of his research in books [(56)~(58)].

I then would like to communicate my deepest gratitude to my late teacher, *Pr. Masato KURIHARA* [(59)~(61)], who cordially taught me the theory of *"Hydrostatics of Ships"* with detailed figures and formulas on the blackboard when I was a 1st year undergraduate student and learned my 1st specialized subject of naval architecture in *the College of Naval Architecture of Nagasaki* in Japan. Therefore, I am following the appearance of my teacher at that time from more than 40 years ago as an exemplary example, when I currently lecture *Hydrostatics of Floating Bodies* and *Theory of Ship Stability* to 2nd year students at my university [(38),(39)].

Finally, let me thank my daughter, *Manami* (White Swiss Shepherd Dog of 5 years old, her certified pedigree's name is *Jewel Manami HORI of Five Stars JP*). The reason is that she is always watching over me while I write a treatise at home.

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